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FREE VIBRATION ANALYSIS OF NON-HOMOGENEOUS RECTANGULAR MEMBRANES USING A HYBRID METHOD

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1. INTRODUCTION

The problem of determining the natural frequencies and mode shapes of membranes is important as a component in the design of many engineering devices. These include microphones, loudspeakers, pumps, compressors, pressure regulators, antennae for space communications, etc. A general review of the dynamic aspects of membranes can be found in review papers by Mazumdar [1]. Recently, there have been many studies of the non-homogeneous membrane whose non-homogeneity is piecewise continuous [2–5]. However, there are few studies on the non-homogeneous membrane whose non-homogeneity is continuous in all domains. Masad [6] solved the problem mentioned above by the finite difference method and the perturbation method. Laura [7] solved the same problem by the optimized Galerkin–Kantrovitch approach and the differential quadrature method.

In this study, a hybrid method composed of differential transforms and the Kantorovitch method is introduced to solve the above-referenced problems. The concept of differential transforms was first proposed by Zhou [8] in 1986 and was applied to solve linear and non-linear initial value problems in electric circuit analysis. Using differential transforms, Chen and Ho [9] proposed a method to solve eigenvalue problems. In this paper, the free vibration problems of a non-homogeneous membrane are considered. Using the Kantorovitch method and the differential transform technique, any natural frequency and the corresponding mode shape can be obtained. Finally, the fundamental natural frequencies and mode shapes of a non-homogeneous membrane are investigated to illustrate the accuracy and efficiency of the present method.

2. DIFFERENTIAL TRANSFORM

In order to solve vibration problems by the hybrid method, the basic theory of differential transform is stated briefly in this section.

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The differential transform of a function $f(x)$ is defined as

$$F(k) = \frac{1}{k!}\left[\frac{\mathrm{d}^k f(x)}{\mathrm{d}x^k}\right]_{x=0}. \tag{1}$$

In equation (1), $f(x)$ is the original function and $F(k)$ is the transformed function, which is called the T-function.

The differential inverse transform of $F(k)$ is defined as

$$f(x) = \sum_{k=0}^{\infty} x^k F(k). \tag{2}$$

From equations (1) and (2), we obtain

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\left[\frac{\mathrm{d}^k f(x)}{\mathrm{d}x^k}\right]_{x=0}. \tag{3}$$

Equations (3) implies that the concept of differential transform is derived from Taylor series expansion. In this study we use the lower-case letter to represent the original function and the upper-case letter to stand for the transformed function (T-function).

From the definitions of equations (1) and (2), it is readily proven that the transformed functions comply with the following basic mathematics operations which are required in this study.

Original function	T-function	
$f(x) = g(x) \pm h(x)$,	$F(k) = G(k) \pm H(k)$,	(4)
$f(x) = cg(x)$,	$F(k) = cG(k)$,	(5)
$f(x) = \frac{\mathrm{d}g(x)}{\mathrm{d}x}$,	$F(k) = (k+1)G(k+1)$,	(6)
$f(x) = \frac{\mathrm{d}g^m(x)}{\mathrm{d}x}$,	$F(k) = (k+1)(k+2) \cdots (k+m)G(k+m)$,	(7)
$f(x) = g(x)h(x)$,	$F(k) = \sum_{r=0}^{k} G(r)H(k-r)$,	(8)
$f(x) = x^m$,	$F(k) = \delta(k-m) = \begin{cases} 1, & k = m, \\ 0, & k \neq m, \end{cases}$	(9)
$f(x) = \sin(\lambda x)$,	$F(k) = \frac{\lambda^k}{k!} \sin\left(\frac{\pi k}{2}\right)$.	(10)

In actual applications, the function $f(x)$ is expressed by a finite series and equation (2) can be written as

$$f(x) = \sum_{k=0}^{n} x^k F(k). \tag{11}$$

Equation (11) implies that $\sum_{k=n+1}^{\infty} x^k F(k)$ is negligibly small. In fact, n is decided by the convergence of natural frequency in this study.

3. USING DIFFERENTIAL TRANSFORM TO ANALYZE THE FREE VIBRATION PROBLEM OF A NON-HOMOGENEOUS RECTANGULAR MEMBRANE

When the non-homogeneous rectangular membrane vibrates in one of its normal modes, the problem is governed by the dimensionless differential equation and boundary conditions as follows:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\omega^2\rho(x,y)u=0, \tag{12}$$

$$u=0, \quad at\ x=0,1, \tag{13}$$

$$u=0 \quad at \quad y=0,\ L\text{-where L is an arbitrary positive constant.} \tag{14}$$

Here, u is the transverse displacement, ω is natural frequency, and $\rho(x, y)$ represents a non-homogeneous factor which is related to the tension and density of the membrane.

By the Kantorovitch method [10, 11] we seek the solutions of the form

$$u(x,y)=f(x)\alpha(y), \tag{15}$$

where $\alpha(y)$ satisfies BC equation (14).

Substituting Eq. (15) into equation (12) results in an error, or residual, function $\varepsilon(x, y)$. Then requiring that

$$\int_0^1\int_0^L \varepsilon(x,y)\alpha(y)\,\mathrm{d}y\,\mathrm{d}x=0, \tag{16}$$

one obtains

$$\int_0^1\{f''+[M+\omega^2N(x)]f\}\mathrm{d}x=0, \tag{17}$$

where

$$M=\frac{\int_0^L\alpha''(y)\alpha(y)\,\mathrm{d}y}{\int_0^L\alpha^2(y)\,\mathrm{d}y}, \quad N(x)=\frac{\int_0^L\rho(x,y)\alpha^2(y)\,\mathrm{d}y}{\int_0^L\alpha^2(y)\,\mathrm{d}y}.$$

For a comparison of previous works [6, 7], we set $\rho(x, y) = \rho(x)$ and $\alpha(y) = \sin(j\pi y/L)$ where $j = 1, 2, 3, \ldots$; then from equation (17), we have

$$f''+\left[\omega^2\rho(x)-\frac{j^2\pi^2}{L^2}\right]f=0. \tag{18}$$

Substituting equation (15) into equation (13), we obtain BCs:

$$f(0)=0, \qquad f(1)=0. \tag{19, 20}$$

For ease of demonstration, set $j = j = \bar{j}$; is a fixed positive integer chosen.

Moreover, taking differential transforms of equation (18) and using equations (4)–(8), we obtain

$$(k+1)(k+2)F(k+2)+\omega^2\sum_{r=0}^{k}P(r)F(k-r)-\frac{\bar{j}^2\pi^2}{L^2}F(k)=0, \tag{21}$$

where $F(k)$ and $P(k)$ are T-functions of $f(x)$ and $\rho(x)$ respectively. Equation (21) is a recursive algebraic equation which is suitable for symbolic computer implementation. Using equation (1), BC equation (19) becomes

$$F(0)=0. \tag{22}$$

Using equation (11), BC equation (20) becomes

$$\sum_{k=0}^{n} F(k) = 0. \tag{23}$$

Equations (22) and (23) are algebraic equations which are suitable for symbolic computer implementation.

Let

$$F(1) = c. \tag{24}$$

Substituting equations (22), (24) and $k = 0$ into equation (21), we have

$$F(2) = 0. \tag{25}$$

Substituting equations (22), (24), (25) and $k = 1$ into equation (21), we have

$$F(3) = c\left(\frac{\bar{j}^2\pi^2}{6L^2} - \frac{P(0)}{6}\,\omega^2\right). \tag{26}$$

Following the same recursive procedure [9], we calculate up to the nth terms $F(n)$ with n being decided by the convergence of the natural frequency as described later.

Substituting $F(0) - F(n)$ into equation (23), we have

$$c[g^{(n)}(\omega)] = 0, \tag{27}$$

where $g^{(n)}(\omega)$ is a polynomial of ω corresponding to n. For non-trivial solutions of mode shapes, we have $c \neq 0$ and

$$g^{(n)}(\omega) = 0. \tag{28}$$

Solving equation (28), we get

$$\omega = \omega_{i\bar{j}}^{(n)} \quad \text{where } i = 1, 2, \ldots . \tag{29}$$

$\omega_{i\bar{j}}^{(n)}$ is the ith estimated natural frequency corresponding to $\bar{j}$ and n, with n being decided by the equation

$$|\omega_{i\bar{j}}^{(n)} - \omega_{i\bar{j}}^{(n-1)}| \leqslant \varepsilon, \tag{30}$$

where $\omega_{i\bar{j}}^{(n-1)}$ is the ith estimated natural frequency corresponding to $\bar{j}$ and $n - 1$ and where ε is a small value set by us. If equation (30) is satisfied, then we have the ith natural frequency corresponding to $\bar{j}$, $\omega_{i\bar{j}} = \omega_{i\bar{j}}^{(n)}$.

Substituting $\omega_{i\bar{j}}$ into $F(0) - F(k)$ and using equation (11), we have

$$f_i(x) = \sum_{k=0}^{n} x^k F^*(k), \tag{31}$$

where $F^*(k)$ is $F(k)$ whose ω is substituted by $\omega_{i\bar{j}}$. Moreover, any ith mode shape of a non-homogeneous membrane corresponding to $\bar{j}$ is obtained as

$$u_{i\bar{j}}(x, y) = f_i(x)\sin\left(\frac{\bar{j}\pi y}{L}\right). \tag{32}$$

Moreover, changing $\bar{j}$ and following the procedure proposed above, all natural frequencies and the corresponding mode shapes of a non-homogeneous membrane can be obtained.

At first glance, the method introduced in this section appears very involved computationally, but actually these algebraic computations, can be finished quickly using symbolic computational software–Mathematics, for example.

4. PROBLEM SOLVING AND RESULTS

To demonstrate the method introduced in this study, two problems are solved here.
Problem 1.

$$\text{GE} \quad f'' + \left[\omega^2(1 + 0{\cdot}1x) - \frac{j^2\pi^2}{L^2}\right]f = 0 \quad \text{where } L = 0{\cdot}2, \tag{33}$$

$$\text{BCs} \quad f(0) = 0, \qquad f(1) = 0. \tag{34, 35}$$

Taking the differential transform of equations (33) and using equation (4)–(9), we obtain

$$(k+1)(k+2)F(k+2) + \omega^2 \sum_{r=0}^{k} [\delta(r) + 0{\cdot}1\delta(r-1)]F(k-r) - \frac{\pi^2}{L^2}F(k) = 0, \tag{36}$$

where we take $j = 1$ for the fundamental natural frequency.

Using equation (1), BC equation (34) becomes

$$F(0) = 0. \tag{37}$$

Using equation (11), BC equation (35) becomes

$$\sum_{k=0}^{n} F(k) = 0. \tag{38}$$

For ease of demonstration, list the computation and results corresponding to $n = 11$.

Let

$$F(1) = c. \tag{39}$$

Substituting equations (37), (39) and $k = 0$ into equation (36), we have

$$F(2) = 0. \tag{40}$$

Substituting equations (37), (39), (40) and $k = 1$ into equation (36), we have

$$F(3) = c(41{\cdot}1234 - 0{\cdot}166667\omega^2). \tag{41}$$

Substituting equations (37), (39)–(41) and $k = 2$ into equation (36), we have

$$F(4) - c(-8{\cdot}33333 \times 10^{-3}\omega). \tag{42}$$

Following the same recursive procedure, we calculate up to the 11th term $F(11)$.

Substituting $F(0) - F(11)$ into equation (38), we have

$$\begin{aligned} g^{(11)}(\omega) &= 36654{\cdot}9 - 680{\cdot}6794\omega^2 + 5{\cdot}04408\omega^4 - 0{\cdot}0186291\omega^6 \\ &\quad + 3{\cdot}42538 \times 10^{-5}\omega^8 - 2{\cdot}50521 \times 10^{-8}\omega^{10} \\ &= 0. \end{aligned} \tag{43}$$

Solving equation (43), we get real roots as follows:

$$\omega = \pm 15{\cdot}6154, \ \pm 16{\cdot}2989, \ \pm 19{\cdot}0854. \tag{44}$$

Considering the fundamental natural frequency, we take

$$\omega_{11}^{(11)} = 15{\cdot}6154. \tag{45}$$

When $n = 10$, by the same way, we obtain

$$\omega_{11}^{(10)} = 15{\cdot}6049. \tag{46}$$

From equations (45) and (46), we have

$$|\omega_{11}^{(11)} - \omega_{11}^{(10)}| = 0.0105 \leqslant \varepsilon, \tag{47}$$

where ε is a small value set by us. If ε is acceptable, from equation (47) we have $\omega_{11} = 15{\cdot}6154$, and ω_{11} is the fundamental natural frequency. Substituting ω_{11} into $F(0) - F(11)$ and using equation (11) we obtain

$$\begin{aligned} f_1(x) = c(x &+ 0{\cdot}483232x^3 - 2{\cdot}03201x^4 + 0{\cdot}070054x^5 \\ &- 0{\cdot}589158x^6 + 1{\cdot}18456x^7 - 0{\cdot}0610072x^8 \\ &+ 0{\cdot}247231x^9 - 0{\cdot}322904x^{10} + 0{\cdot}0200402x^{11}). \end{aligned} \tag{48}$$

Substituting equation (48) and $\bar{j} = 1$ into equation (32), we obtain the fundamental mode shape as

$$u_{11}(x, y) = f_1(x) \sin\left(\frac{\pi y}{L}\right). \tag{49}$$

The convergence of the fundamental natural frequency is shown in Figure 1 where ω_{11} converges to $15{\cdot}61333$ ($n = 20$, $\varepsilon = 0$). Moreover, the comparison between cross-sections of mode shapes (along the x-axis, at $y = L/2$) of a homogeneous membrane and a non-homogeneous membrane are shown in Figure 2. It is clear that the inhomogeneity shifts the mode shape peak towards regions of higher mass concentration. Finally, values of the fundamental natural frequency ($\varepsilon = 0$) are given in Table 1. From the comparison between the present results and those of previous work [6, 7] in Table 1, excellent agreement is observed.

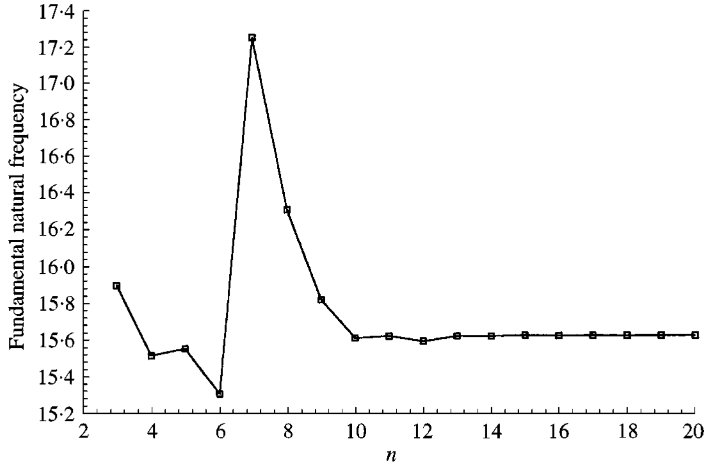

Figure 1. The convergence of the fundamental natural frequency of a non-homogeneous membrane with $\rho(x) = 1 + 0{\cdot}1x$.

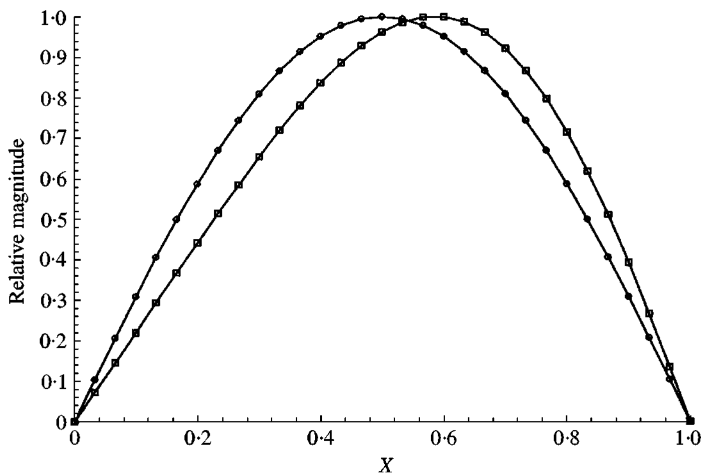

Figure 2. A cross-section of the fundamental mode shape for $\rho(x) = 1 + 0{\cdot}1x$, along the x-axis, at $y = L/2$: —○— homogeneous; —□— non-homogeneous.

TABLE 1

The fundamental natural frequencies of homogeneous (H) membrane and non-homogeneous (NH) membrane with $\rho(x) = 1 + 0{\cdot}1x$

L	H	NH [6]	NH [7]	NH [present]; $\varepsilon = 0$
1·0	4·44288	4·33538	4·33539	4·33538 ($n = 17$)
0·8	5·02900	4·90719	4·90719	4·90719 ($n = 17$)
0·6	6·10616	5·95790	5·95790	5·95790 ($n = 17$)
0·4	8·45900	8·25220	8·25221	8·25220 ($n = 18$)
0·2	16·0190	15·6133	15·61334	15·61333 ($n = 20$)

Problem 2.

$$\text{GE} \quad f'' + \left[\omega^2(1 + 0{\cdot}1 \sin \pi x) - \frac{j^2\pi^2}{L^2}\right] f = 0 \quad \text{where } L = 0{\cdot}2, \tag{50}$$

$$\text{BCs} \quad f(0) = 0, \qquad f(1) = 0. \tag{51, 52}$$

Taking differential transform of equation (50) and using equations (4)–(10), we obtain

$$(k + 1)(k + 2)F(k + 2) + \omega^2 \sum_{r=0}^{k} \left[\delta(r) + 0{\cdot}1 \frac{\pi^r}{r!} \sin\left(\frac{\pi r}{2}\right)\right] F(k - r) \quad - \frac{\pi^2}{L^2} F(k) = 0, \tag{53}$$

where we take $j = 1$ for the fundamental natural frequency. Using equation (53) and by the same way as described in Problem 1, we obtain results as follows. The convergence of the fundamental natural frequency is shown in Figure 3 where ω_{11} converges to 15·37381 ($n = 56$, $\varepsilon = 0$). Moreover the comparison between cross-sections of mode shapes (along the x-axis, at $y = L/2$) of a homogeneous membrane and a non-homogeneous membrane are

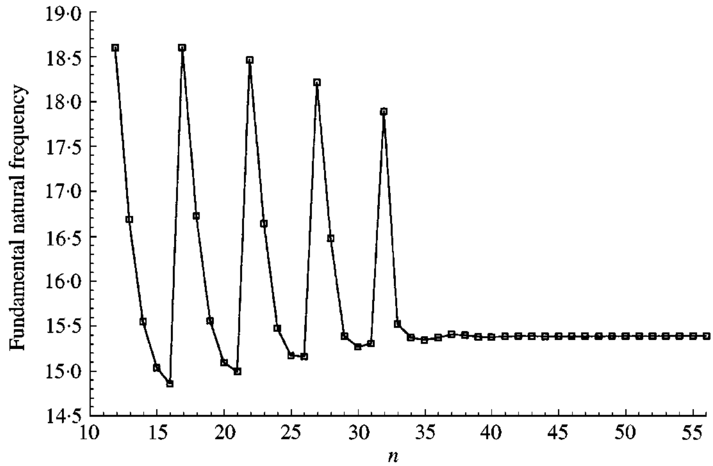

Figure 3. The convergence of the fundamental natural frequency of a non-homogeneous membrane with $\rho(x) = 1 + 0{\cdot}1 \sin \pi X$.

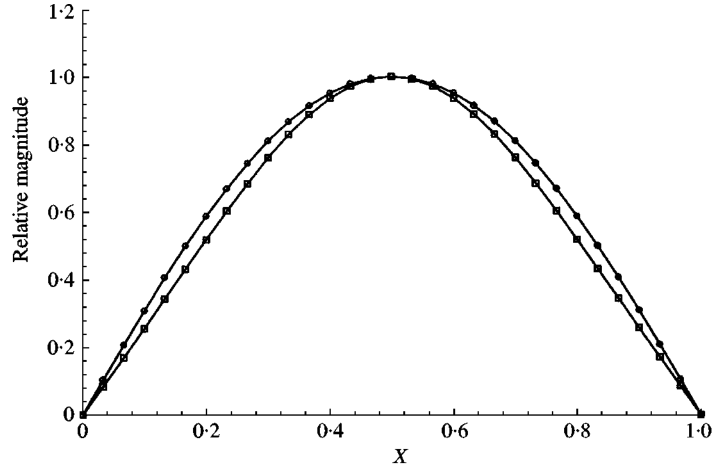

Figure 4. A cross-section of the fundamental mode shape for $\rho(x) = 1 + 0{\cdot}1 \sin \pi X$, along the x-axis and at $y = L/2$: —○— homogeneous; —□— non-homogeneous.

TABLE 2

The fundamental natural frequencies of homogeneous (H) membrane and non-homogeneous (NH) membrane with $\rho(x) = 1 + 0{\cdot}1 \sin \pi x$

L	H	NH [present]; $\varepsilon = 0$
1·0	4·44288	4·26541 ($n = 30$)
0·8	5·02900	4·82806 ($n = 32$)
0·6	6·10616	5·86207 ($n = 35$)
0·4	8·45900	8·12044 ($n = 43$)
0·2	16·0190	15·37381 ($n = 56$)

shown in Figure 4. It is clear that the inhomogeneity shifts both sides of the mode shape towards the inside. Finally, values of the fundamental natural frequency ($\varepsilon = 0$) are given in Table 2.

CONCLUSION

In summary, using the hybrid method to solve the free vibration problems of a non-homogeneous rectangular membrane consists of four main steps. The steps are using the Kantorovitch method to get an ODE with variable coefficient, transforming the ODE into algebraic equations, solving the equations, and inverting the solution of the equations to obtain any natural frequency and the corresponding mode shape. The calculated results are highly compatible with those of previous studies on this subject.

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